Accessibility in transport modelling - The 4S model

Paper

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Abstract

Cities provide a range of opportunities to its residents, from workplaces to schools, shops and entertainment. However taking advantage of these opportunities requires some degree of travel, which may limit their ability to take full advantage of these opportunities. The key analytical tool that is used to explore the quality and ease of access to opportunities is **accessibility**. This paper explains in detail the most established definitions of accessibility and a new approach used by TransPosition's multi-model strategic model, the 4S model.

1 Introduction

Accessibility is highly correlated to the liveability of an area and is related to the opportunities available to people and the ease of accessing them using all modes of transport. As such it provides a useful measure to asses a cities performance. Historically there have been two key ways to assessing accessibility, Hansen accessibility measure and Log-Sum accessibility. The first approach focuses on what can be reached and the second focuses on costs or travel time. These two methods are explained in detail in sec. 2.

The 4S model, allows the accessibility across all modes of transport, walking, cycling, car and public transport, to be determined simultaneously. The model uses a generalised utility function with a specific component for the utility attained from the attraction location itself, due to the nature of the 4S model it is very easy to prepare an accessibility measure. Here the accessibility measure is simply the average utility that can be attained at a location, averaged across all Monte Carlo draws. The utility structure of the 4S model is explained in detail in sec. 3.

2 Definitions of Accessibility

Historically there have been two key ways of assessing accessibility. The first approach focuses on what can be reached and the second focuses on costs or travel time.

2.1 Hansen accessibility measures

Some of the earliest accessibility work was done by Hanson in 1959 (Hansen (1959)). He defined accessibility as "the potential of opportunities for interaction" and his index assumed that the accessibility to an activity is directly proportional to the size of the activity and inversely proportional to some function of the distance/time/cost to the location of the activity. Hansen explored power functions, and noted that the exponent should be lower for work (0.9) and higher for purposes where the specific destination makes less of a difference (2.0 for shopping, for example).

In its generalised form the Hansen accessibility index is described by the following equation.

$$a_i = \sum_j s_i f(c_{ij})$$

where

- a_i is the accessibility for zone i for a given market segment/travel purpose
- s_i is the measure of opportunities available for that market segment in zone j (measured in units such as persons, jobs, GFA etc)

- $f()$ is a deterrence function, similar to that used in gravity models. Hansen used c_{ij}^{-n} but more recent models tend to use $e^{-kc_{ij}}$
- c_{ij} is the measure of separation between zones i and j – could be distance, travel time or generalised cost

This factor is sometimes normalised using the systemwide total of S - i.e. $a_i^* = \frac{a_i}{\sum_j s_i}$

This approach can be easily extended to other modes such as seen in the WalkScore website which prepares a score based on the weighted activities that can be reached (Duncan et al. (2011)) by walking or public transport, or in Queensland Transport's LUPTAI model (Pitot et al. (2006), Bertolaccini et al. (2017)).

An even simple approach that is sometimes used is to report the number of opportunities that can be reached within a certain travel time - for example there are x jobs within 30 minutes of a location. This is simply as special case of the Hansen measure where

$$f(c_{ij}) = \begin{cases} 1 & if c_{ij} \leq C \\ 0 & if c_{ij} > C \end{cases}$$

The difficulty with these approaches is that travellers are not so much interested in the number of activities that they can reach but the value that we get from those destinations. The effect of size on value is non-linear as we would expect a diminishing return to increased number of opportunities. A bigger shopping centre is better than a smaller one, and having more choices is (usually) better than having fewer ones. But the extra value offered by each new alternative is always decreasing - there is less and less chance that it will offer something new.

It is possible to adjust the weighted size measures to account for this diminishing return. The AAM model developed by ARRB (Espada and Luk (2011)) includes a saturation function, such that the score is 0 when no activities can be reached and 1 once a threshold summed opportunities is available.

But a simpler transformation was presented in Patton (1970) and expanded on in Davidson (1977). Noting that the Hansen Accessibility index is a sum of opportunities weighted by a cost function, we can postulate a special cost (called the centrality) that would give the same index when applied to the unweighted sum of opportunities.

$$a_i = \sum_j s_i f(c_{ij}) = \left(\sum_j s_i\right) f(Y_i)$$

so

$$y_i = f^{-1}\left(\frac{\sum_j s_i f(c_{ij}))}{\sum_j s_i}\right)$$

If $f(x) = e^{-\beta x}$ then $f^{-1}(x) = -\frac{1}{\beta}\ln(x)$ and $S_T = \sum_j s_i$ we get

$$y_i = -\frac{1}{\beta} ln\left(\sum_j s_i e^{-\beta c_{ij}}\right) + K \qquad (1)$$

Where K is a constant for a given total activity size ($= \frac{1}{\beta} ln(S_T)$). This formulation converts the weighted size measure into a cost measure. Furthermore it has an intuitive conceptualization – if all of the cities activities were located at a single point, how far would we be from that point.

Finally the form of this equation would be familiar to anyone who has studied discrete choice analysis - it is the logsum. Which leads neatly into the second major approach to accessibility.

2.2 Log-Sum Accessibility

Arguably the biggest theoretical advance in transport modelling came from the work done by Daniel McFadden on random utility models (RUM). These were focused on how people made decisions between discrete choices when their perceptions had variability and the modellers are uncertain about their valuations of the alternatives. The most generalised model assumes that when making decisions people mentally ascribe a value to each alternative (the utility) and then chooses the one with the highest utility. Since there is variation in these utilities (either due to uncertainty or variability) they can all be described by random variables. The probability of choosing alternative i is simply the probability that its utility is greater than any of the others.

The simplest formulation of the RUM is the logit model (for a good discussion of this see Train (2009) ch. 3). In this model the random variables are significantly simplified. Each alternative is assumed to have a fixed component and a variable component, and furthermore the variable component is identically and independently distributed (IID). That means that each alternative's utility has a random component added to it, and these random components are all taken from the same distribution, and each of them is chosen independently.

$$U_i = v_i + \varepsilon_i$$

where v_i is the fixed component to the random utility U_i and ε is the error term. In the logit model the error term is assumed to be a type I extreme value (pdf is $f(\varepsilon) = e^{-\varepsilon}e^{-e^{-\varepsilon}}$).

Under these assumptions, the probability of choosing alternative i is simply the probability that its error term will be greater than any other error term minus the difference in the fixed components. i.e.

$$P(i) = P(\varepsilon_j < \varepsilon_i - (V_j - V_i)\forall j \neq i) = \frac{e^{V_i}}{\sum_j e^{V_j}} = \frac{e^{\beta x_i}}{\sum_j e^{\beta x_j}}$$

Where x_j is a vector of attributes that describe alternative j (such as time, cost etc) and β is a scaling parameter that is generally chosen such that the utility values can be measured in monetary units.

It can be shown (Williams (1977), Small and Rosen (1981)) that by adding an additional assumption (that the marginal utility of income is constant for each person, at least over the range of changes being considered) it is possible to derive a measure of consumer surplus from this logit expression. It is calculated as the expected value of utility given that people always choose the alternative that gives them the highest utility.

$$E(CS) = \frac{1}{\beta}E\left[max_j(V_j + \varepsilon_j)\right] + C = \frac{1}{\beta}ln\left(\sum_j e^{V_j}\right) + C \qquad (2)$$

It should be noted that utility formulations have no fixed zero point, since they are only comparative and any constant can be added to all alternatives without changing the choice. There is also no fixed scale, but the β value is chosen to ensure that the utility values are in dollars. The C value at the end of eq. 2 is an unknown constant that "represents the fact that the absolute level of utility cannot be measured" – Train (2009) pg55.

When applied to destination choice the utility function $V_{0,j}$ denotes the discrete choice utility for travel from the origin o to destination j. It must have a component that includes a function of the size of the destination $(f(s_j)$ and a vector of components that describe the cost of travelling to the destination $(\mathbf{x_{oj}})$ and a vector of coefficients β'. For simplicity we will assume a generalised cost measured in dollars – any component of time/distance/cost can be included with appropriate factors. The coefficients are normalised by dividing by a fixed term β such that the resulting coefficients in β' associated with cost terms are equal to 1.

$$V_{oj} = \beta f(s_j) - \boldsymbol{\beta}' \cdot \mathbf{x}_{oj} \qquad (3)$$

In destination choice models the alternatives are usually grouped into zones, and so the size of a destination is actually the sum of sizes of all of the destinations in that zone. In order for the model not to be dependent on the specifics of the zoning system we must ensure that the probability of travelling to a zone does not change if the zoning system is rearranged. So if zone j is split into two zones $j1$ and $j2$ with sizes s_{j1} and s_{j2} then $p(j) = p(j1) + p(j2)$. With the logit model this is true only if the utility increases with the logarithm of size. So the final consumer surplus equation is

$$E(CS) = -\frac{1}{\beta} ln \left(\sum_j e^{\beta ln(s_i) - c_{ij}} \right) + C = -\frac{1}{\beta} ln \left(\sum_j s_i e^{-\beta c_{ij}} \right) + C \qquad (4)$$

It can be seen that this equation is identical in form to that shown in eq. 1. This comparison gives a number of insights. Firstly the centrality measure given in eq. 1 is a consumer surplus measure, at least for a fixed total population size. By equating K in eq. 1 with C in eq. 2 we can see that it is possible to choose a value for C that is not completely arbitrary. A natural value for C is the utility of the whole city $\frac{1}{\beta} ln(S_T)$. The consumer surplus can then be understood as having two components – the total utility that would be available from the city if all travel was free and instantaneous, and the reduction in that utility due to the actual time/cost of travel. The centrality measure reflects (in units of cost) how far we are from the ideal situation.

3 The 4S Model

3.1 Utility structure

The 4S model is a different approach to transport modelling developed by the author and applied to various Australian cities over the last 8 years. A full discussion of the model is beyond the scope of this paper – more details can be found in (Davidson 2011, 2017). This paper will focus on the attraction utility/destination choice aspects of the model and implications for the calculation of accessibility using the 4S approach.

The Segmented Stochastic Slice Simulation (4S) Model is based on a new algorithm that can efficiently solve a generalised utility function where all elements are random variables. Unlike the more common four-step models, which consider each element of travel separately (trip generation, trip distribution, mode choice, assignment), the 4S model considers all elements simultaneously as part of the multi-component utility function.

The model breaks down the utility of travel choice into two parts - the utility of the attractor and the disutilities of travelling.

$$U_{a,m,r,n} = U_{a,n} - \boldsymbol{\beta}_n \cdot \mathbf{C}_{a,m,r,n} \qquad (5)$$

where $U_{a,n}$ is the intrinsic utility of the attractor *a* to the individual *n*, and $\mathbf{C}_{a,m,r,n}$ is the vector of cost components of travelling to attractor *a* by mode *m* on route *r*, $\boldsymbol{\beta}_n$ is a vector of random taste coefficients for individual *n*. The coefficients are random variables that vary over individual decision makers and are distributed with a density function that is described by a distribution (such as normal, uniform, triangular, gamma, log-normal) and parameters (such as mean and standard deviation) of β's across the market segment under consideration. With suitable constraints on model formulation, these parameters may be estimated using maximum likelihood methods using similar datasets as those used for calibrating traditional destination and mode choice models.

Thus the trips between a particular production location *p* and an attractor *i*, by a particular mode *m* and route *r* for a market segment *s* is dependent on the size of the market segment at the production $S_{p,s}$ and the probability that the attractor, mode, route combination is optimal.

$$T_{p,a,m,r,s} = S_{p,s}\, p(U_{i,m,r,s} > U_{j,m,r,s},\ \forall j \in A_s,\ j \neq i) \qquad (6)$$

Where A_s is the set of all possible attractors for market segment *s*.

It can be seen that this structure is a generalisation of the fundamental utility function that lies at the heart of random utility modelling, since all elements of the equation can be independently specified random variables with no constraints on their distributions. In that sense it is similar to the mixed logit model, which combines a generalised utility function with the traditional logit error distribution. The difference is that the 4S model does not require the logit error distribution (although it could be included if desired).

The algorithm does put one constraint on the overall structure of the utility function – all positive utilities (benefits) must be allocated at the attractor and all other utility components must be negative (costs). Thus the positive utilities are inherent in each location, and can be a function of any aspect of that location and market segment (including the traditional size measures such as jobs, GFA, population etc as well as more qualitative parameters). The negative utilities are associated with the network, and depend on any aspects of the route and modes taken.

3.2 Algorithmic approach

The model can find solutions for the generalised utility function through a combination of two techniques. The first is Monte Carlo simulation. This makes use of pseudo-random numbers to probe the various probability distributions – rather than trying to find an analytical solution to the complex integration required in the calculation of discrete choice probabilities, random samples are taken and aggregated. If enough samples are taken then the mean of the sampled values will approximate the true solution of the integral. This is the standard approach for distributions that do not have a simple closed form, including the mixed logit model.

However the applications of mixed logit choice models still requires the enumeration of all choice alternatives. So for a classic destination choice model, for example, the destinations are grouped into traffic zones (TAZ) and costs are calculated between every combination of zones – for simple logit models this is done with one or more skim matrices (often joined using a logit mode choice model). This is necessary because all alternative choices must be enumerated. If the choice between destination, mode and route is combined then the full list of possible discrete alternatives is impossibly large. Which leads to the second core technique of the 4S model – the maximum utility slice. Since at heart the RUM model is about finding the best option amongst a set of discrete choices, fully enumerating all choices is inefficient if the best one can be found without enumeration. It turns out that as long as the utility equation is structured in the form given above, where the only positive utility component occurs at the attractor, a modification of the standard shortest path algorithm can be used to identify the single best alternative for each location. The existence of a single best alternative is true for a slice, since the random variation in the utility functions is dealt with separately in the Monte Carlo component. A single 'slice' of the model works with a single set of coefficients drawn from each of the probability distributions that are included in the utility function, but applies these to decision makers at every location in the network simultaneously. By working with single values for all parameters, the problem becomes a straightforward optimisation problem. The model uses an agent simulation approach, but rather than modelling a single agent at a time, the slice effectively allows for an agent "cloud".

As it has been implemented, the model uses a single multi-modal network, where mode choice is treated as a component of route choice. The advantage of this approach is that complex mode structures can be included without any additional effort. This removes one of the simplifications inherent in most four-step models, where each mode combination is treated separately (e.g. car, walk, PT with walk access, PT with park-n-ride access etc). In the 4S model these would all be included in the single network, and any given journey can make use of any mode combination that makes sense.

3.3 Attraction utility

As discussed above, the 4S model does not require full enumeration of all alternatives – destinations, modes and routes are considered only if they are feasible for providing the highest utility. Any alternative that is clearly not the best will be rejected, pruning away a vast set of feasible but suboptimal alternatives. This feature removes the requirement for aggregating travel into zones – the set of feasible destinations for the choice model can include every point in the network, not just zone centroids. In this way all travel is from point to point rather than from zone to zone. The model could allow travel to arbitrary locations, and we have explored an approach where cadastral boundaries are used to identify individual lots for possible destinations. However because of the way that the transport network is structured it is only possible to make meaningful choices at network nodes – for points along a link all feasible choices can be captured by the choices at either end of the node. This allows the destination set to be simplified to include only network nodes.

In order for this process to work all activities (jobs, people, shops etc) must be located at network nodes. The utility for a particular market segment of engaging in activities at that node must then be calculated. So, for example, if the model is considering a slice of home-based-shopping travel, the set of feasible destinations would be all locations that have some level of retail employment (or retail GFA). Within the model every node would have a size value for retail employment – most nodes would have zero, and some would have large numbers if they are the nodes adjacent to a major shopping centre.

The allocation of demographic and land use variables to zones is done automatically within the model using a whole series of heuristics and allocation rules. The model allows for some land use types to be specified as points or polygons, representing individual schools, hospitals or shopping centres (for example). For these types the activity will typically be allocated to one or more nodes immediately within or adjacent to that location. Other data, such as population data taken from the ABS Census, will be more aggregate, usually at Australian Statistical Geography Standard (ASGS) SA1 or SA2 levels. This data is typically allocated to most of the nodes within that boundary, with the share allocated to each node a function of the details of the node type and its adjacent links (for example we usually do not allocate jobs to nodes that have no car access).

Because the demographic and land use variables are so spatially disaggregate, there is a need to be careful about the issues raised above in the discussion about eq. 3 regarding the need for consistency in aggregate and disaggrgate probability. In general terms we need to ensure that the probability choosing any one of a cluster of destinations should be equal to the probability of travelling to a single destination with values equal to the sum of the clusters. In the previous discussion the goal was to ensure that changes to zoning systems do not change model outcomes, here it is the need to be invariant under different node-demographic allocations.

It was noted above that under a logit model the assumption that utility increased with the logarithm of size was sufficient to guarantee consistency under different zone aggregation schemes. But for the 4S model we need not a single value for the attraction utility, but a full probability distribution. We know that it needs to have the same qualities as a logarithm, in the sense that there should be diminishing returns with size and aggregation consistency. The attraction utility distribution also needs to satisfy the requirements of explaining the variations in destination choice, where people are more likely to travel to large, close destinations that smaller, further away ones. Ideally the attraction utility functions should have some satisfying behavioural interpretation as well.

The approach that has been adopted in the 4S model is to assume that any size measure is correlated with a count of opportunities. Each opportunity is then given a fundamental utility distribution, reflecting the likelihood that a given opportunity will be satisfactory. For example, if we are wanting to buy some shoes then we would consider all of the locations that sell shoes. We might use the number of employees at the establishment as the size measure, or perhaps the floor area of the shop. We might assume that every job, or every $10m^2$ of floor space provides us with an opportunity to find the shoes that we want. A shop that is five times larger will give five times as many opportunities to be satisfied. When making a single choice (a single "slice" within the model) we will draw values from the fundamental distribution for each separate opportunity. The utility of a particular destination is then the highest drawn value from all of the opportunities at that location. The shop that is five times bigger will have five times as many opportunities at satisfying us, but in the end is only as valuable as its highest utility opportunity.

$$U_a, m = Max_{i=1..n_m}(\mathbf{u_m})$$

where n_m is the number of opportunities at attractor a for market segment m and $\mathbf{u_m}$ is the underlying opportunity distribution for that market segment.

It can be seen that this formulation implicitly satisfies both of the aforementioned characteristics. The utility has a diminishing return with size because there is a decreasing chance that the marginal opportunity will exceed the best of the preceding opportunities. And the aggregation is ensured because the highest utility of a cluster of destinations will equal (probabilistically) the highest utility of a single destination with the aggregate number of opportunities.

It is important for the discussion on accessibility below to note that this aggregation can be done up to any level – up to and including a whole city. For any given traveller in a given market segment, the maximum possible utility of the whole city is the value of the most attractive opportunity in the city, without consideration of travel costs. This will tend to increase with the logarithm of the city size and any actual expected utility will be less that this due to the fact that the opportunities are spatially dispersed and travel costs are not zero.

The underlying probability distribution for a single opportunity could take a number of possible forms. There are good reasons to think that a "fat-tailed" distribution would be most appropriate, as we can observe that for most travel choices there are a vast number of unsatisfactory choices with low utility and a small number of high utility options. For example, when travelling to work most people will travel past a huge number of jobs (in large cities it may be millions) to reach the one that is appropriate for them. For the home-to-work travel the variation in the underlying utility is large, but most alternatives would be unattractive with a low utility.

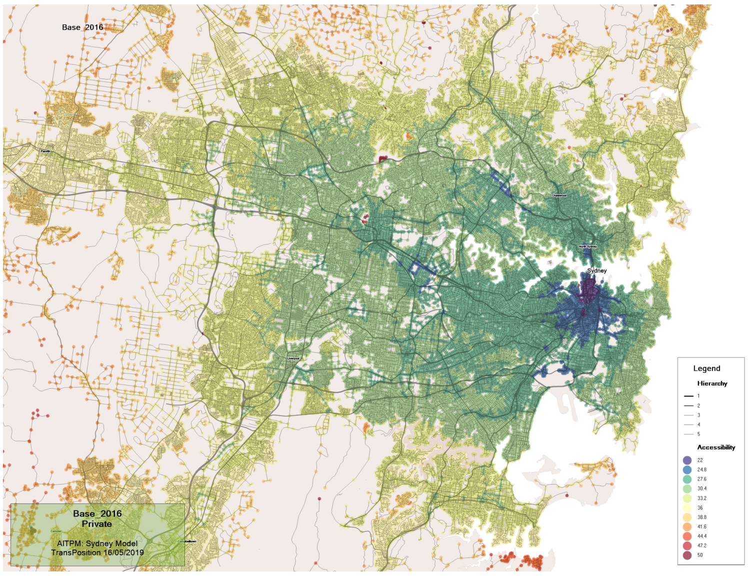
It should be noted that the distribution of the underlying opportunity utility is a measure of how much variation there is in people's assessment of alternative destinations, and is also correlated with average trip length. If we want to go to the shops to buy bread and milk then any shop that has this is pretty similar – we will probably just go to the closest one. This is reflected by a opportunity utility with a very low standard deviation and, depending on the arrangement of opportunities, a short trip length. As discussed above, work travel will tend to have a large deviation and high average trip lengths. There are some activities that may have small variation but large trip lengths due to few choices – an example might be container freight travelling through ports. Each port may have similar inherent attractiveness with low variation, but average trip lengths will be long due to the small number of ports.

For flexibility we have chosen to use a gamma distribution for the underlying opportunity utility.

3.4 Accessibility in the 4S Model

Since the 4S model uses a generalised utility function with a specific component for the utility attained from the attraction location itself (the attraction utility), it is very easy to prepare an accessibility measure. This is simply the average utility that can be attained at a location, averaged across all Monte Carlo draws. The accessibility measure can also be calculated for specific market segments by averaging only those slices that were done for travellers in that market segment. Since the utility function used by the model is always specified in cost terms the average utility is a direct consumer surplus measure in dollars, but with an unknown constant C (similar to that included in eq. 2). Plotting this utility value the accessibility profile of an entire city can be obtained. For example fig. 1 shows the accessibility of Sydney, highlighting the dividing effect of the harbour, with relatively higher accessibilities on the southern side.

Figure 1: Accessibility of Sydney for the base model



As the average utility value is a direct cost with an unknown constant, the differences in utility can be easily calculated since the constant term cancels out. Thus differences in utility can be used in project evaluation, or to understand the winners and losers associated with any change to the system. By plotting these differences it is possible to easily understand the spatial accessibility impacts of a project/policy. An examples of this "benefit footprint" map is shown in fig. 2 which highlights the negative effect of allowing only PT and active transport modes and fig. 3 which shows the benefit of taxis when only PT and active transport modes are available.

Figure 2: Change to Sydney's accessibility when only PT and active transport modes are available

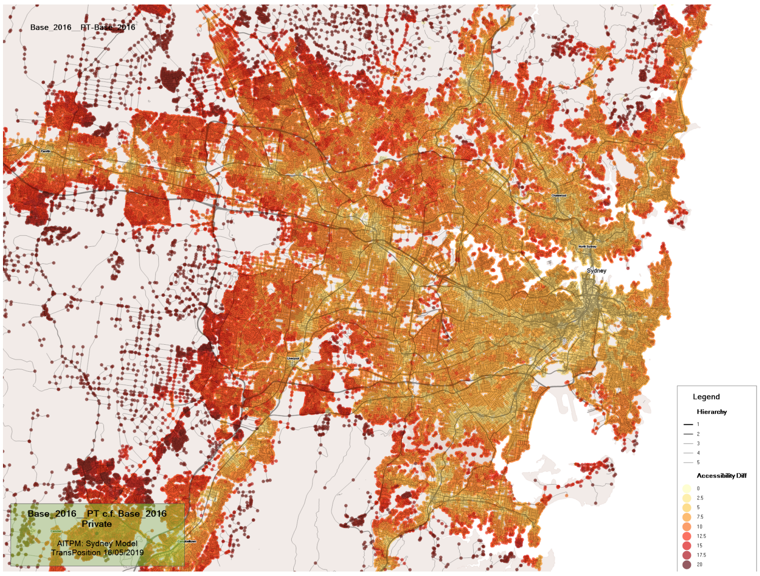
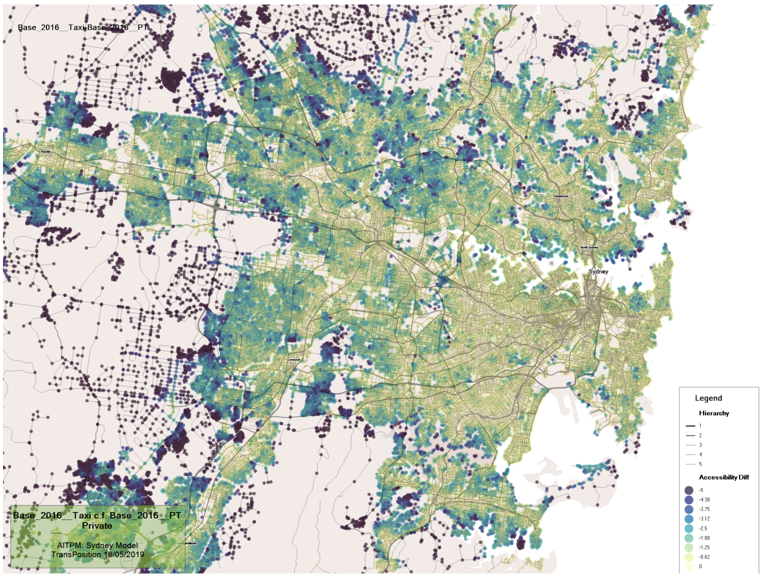


Figure 3: Change to Sydney's accessibility when taxis are available in addition to the PT and Active transport modes



However it would also be useful to find a way to standardise the accessibility outputs so that they can be more easily understood and compared. The interpretation of eq. 1 through the insights of eq. 4 suggest a useful approach. As discussed above, the centrality measure can be understood as the hypothetical cost of travel to the aggregate opportunities available in a city. The total utility is then the utility available from the whole city if it were all at one place, minus the cost of travelling to that one place.

Interestingly, this also gives a way of understanding the Hansen accessibility measure – it is the effective size of the city at that location, assuming that nothing else exists but travel is instantaneous. So, taking accessibility to population as an example, if we had a city with 1,000,000 people then the utility available if we had instantaneous access to all of them might be calculated (using the attraction utility function) as being equal to \$73+C. The calculated utility at a particular location might be equal to

$52+C. This would mean that the centrality would be $21 (with the C's cancelling out) because we are $21 worse off than we would be if travel to all of the locations in the city were instantaneous. And the Hansen equivalent population might be 60,000 – since this is the aggregate population that would give the utility of $52 if all travel were instantaneous. This also highlights the problems with the raw Hansen index in that it is a non-linear measure of accessibility, due to the logarithmic form of the attraction utility function.

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